MATH 2010E TUTO 7

(1) Let γ be the (parametrized) curve given by

$$\gamma(t) = (t^2, \frac{t^3}{3} - t),\ t \in [-\sqrt{3}, \sqrt{3}].$$

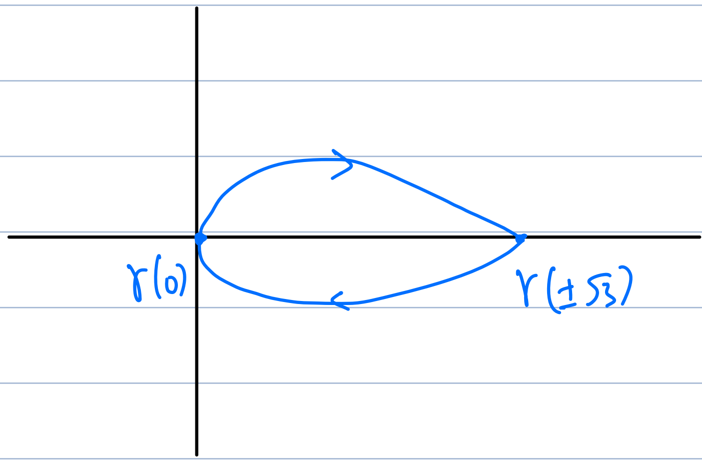
(a) (5 points) Sketch the curve γ.

(b) (5 points) Show that it is a simple closed curve.

(c) (2 points) Calculate the tangent vector of γ at t for all $t \in (-\sqrt{3}, \sqrt{3})$.

(d) (4 points) Calculate the arc-length of γ.

Ans: a)



b) $\gamma(\sqrt{3}) = \gamma(-\sqrt{3}) \Rightarrow \gamma$ is closed

Suppose $t_1, t_2 \in (-\sqrt{3}, \sqrt{3})$ s.t. $t_1 \neq t_2$ and $\gamma(t_1) = \gamma(t_2)$

Then $t_1^2 = t_2^2$ and $\frac{1}{3}t_1^3 - t_1 = \frac{1}{3}t_2^3 - t_2$

$\Rightarrow t_1 = \pm t_2$

$\Rightarrow t_1 = -t_2$, then $\frac{1}{3}t_2^3 - t_2 = 0$

$\Rightarrow t_2 = 0$ since $t_2 \neq \pm\sqrt{3}$

Hence $t_1 = t_2 = 0$, contradiction

So γ is simple.

c) $\gamma'(t) = (2t, t^2 - 1)$

d) arc length $= \int_{-\sqrt{3}}^{\sqrt{3}} \|\gamma'(t)\|\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (t^2-1)^2}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} (t^2+1)\, dt$

(2) Let P be the plane passing through the points

$$(1,-1,1),\ (2,2,0),\ \text{and}\ (0,1,-2).$$

(a) (4 points) Find a normal vector to P.

(b) (4 points) Find an equation of P of the form

$$ax+by+cz=d,$$

where a, b, c, d are constants.

(c) (2 points) Find a parametrization of the line passing through the point $(0,0,1)$ and perpendicular to P in the following form

$$\vec{x}=\vec{a}+t\vec{v},$$

for some $\vec{a},\vec{v}\in\mathbb{R}^3$.

(d) (6 points) Using your result in previous parts, find the exact distance from the point $(0,0,1)$ to the plane P.

Ans: a) Let $\vec{u} = (1,-1,1) - (0,1,-2) = (1,-2,3)$

$\vec{v} = (2,2,0) - (0,1,-2) = (2,1,2)$

Then $\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ 2 & 1 & 2 \end{vmatrix}$

$= (-7,4,5)$ is normal to P.

b) By a) the eqn of P has the form

$-7x+4y+5z=d \quad (*)$

Put $(1,-1,1)$ into $(*)$,

$d=-6$

The eqn of P is $-7x+4y+5z=-6$

(c) (2 points) Find a parametrization of the line passing through the point $(0,0,1)$ and perpendicular to P in the following form

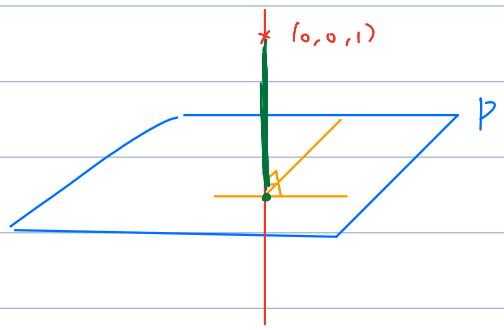
$$\vec{x} = \vec{a} + t\vec{v},$$

for some $\vec{a}, \vec{v} \in \mathbb{R}^3$.

(d) (6 points) Using your result in previous parts, find the exact distance from the point $(0,0,1)$ to the plane P.

c) $\vec{x} = (0,0,1) + t(-7,4,5)$

d)



Put $\vec{x} = (0,0,1) + t(-7,4,5)$

into $-7x + 4y + 5z = -6$

Then $49t + 16t + 5(5t+1) = -6$

$$90t = -11$$

$$t = \frac{-11}{90}$$

Intersection pt is $\left(\frac{77}{90}, \frac{-44}{90}, \frac{35}{90}\right)$

Distance $= \sqrt{\left(\frac{77}{90} - 0\right)^2 + \left(-\frac{44}{90} - 0\right)^2 + \left(\frac{35}{90} - 1\right)^2}$

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(3) (a) (4 points) Find the limit of $f(x,y) = \dfrac{2xy}{x^2+y^2}$ as $(x,y) \to (0,0)$ along the line with slope m.

(b) (4 points) Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist? Justify your answer.

(c) (4 points) Find the limit of $g(x,y) = \dfrac{2x^2y}{x^4+y^2}$ as $(x,y) \to (0,0)$ along the line with slope m.

(d) (4 points) Does $\lim\limits_{(x,y)\to(0,0)} g(x,y)$ exist? Justify your answer.

a) Along $y = mx$,

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2m}{1+m^2}$$

b) Different limits along different paths (different m)

$\Rightarrow$ limit DNE

c) Along $y = mx$

$$\lim_{x\to 0} g(x, mx) = \lim_{x\to 0} \frac{2x^2(mx)}{x^4 + (mx)^2} = \lim_{x\to 0} \frac{2m\cancel{x^3}^{\,x}}{\cancel{x^4}_{\,x^2} + m\cancel{x^2}_{\,1}} = 0$$

d) Along $y = kx^2$

$$\lim_{x\to 0} g(x, kx^2) = \lim_{x\to 0} \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2k}{1+k^2}$$

Different k, different limits

$\Rightarrow$ limit DNE

(4) Consider rational function

$$f(x,y) = \frac{2x^2 + 3x^2y^2 + 2y^2}{x^2 + y^2}$$

(a) (2 points) Let A be the domain of definition of f, find A.

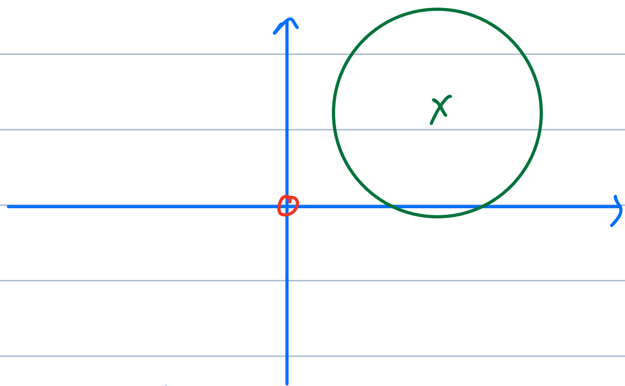
(b) (6 points) Find, without justification, the interior $\mathrm{Int}(A)$, exterior $\mathrm{Ext}(A)$, and boundary ∂A of A.

(c) (4 points) Is A open? Is A closed? Justify your answers.

(d) (4 points) Can f be extended to a function continuous on $\mathbb{R}^2$? Justify your answer and, if so, find the value of the extended function $f(\vec{a})$ at those $\vec{a} \notin A$.

Ans: a) $A = \mathbb{R}^2 \setminus \{x^2+y^2=0\} = \mathbb{R}^2 \setminus \{(0,0)\}$.

b)



$\mathrm{Int}(A) = A$

$\mathrm{Ext}(A) = \mathrm{Int}(\{(0,0)\}) = \phi$

$\partial A = \{(0,0)\}$

c) A is open since $\mathrm{Int}(A) = A$

A is not closed since $\mathbb{R}^2 \setminus A = \{(0,0)\}$ is not open

(4) Consider rational function

$$f(x,y) = \frac{2x^2 + 3x^2y^2 + 2y^2}{x^2 + y^2}$$

(a) (2 points) Let A be the domain of definition of f, find A.

(b) (6 points) Find, without justification, the interior $\mathrm{Int}(A)$, exterior $\mathrm{Ext}(A)$, and boundary ∂A of A.

(c) (4 points) Is A open? Is A closed? Justify your answers.

(d) (4 points) Can f be extended to a function continuous on $\mathbb{R}^2$? Justify your answer and, if so, find the value of the extended function $f(\vec{a})$ at those $\vec{a} \notin A$.

d) $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{2r^2\cos^2\theta + 3r^2\cos^2\theta \cdot r^2\sin^2\theta + 2r^2\sin^2\theta}{r^2}$$

$$= \lim_{r\to 0} \left(2 + 3r^2 \underline{\cos^2\theta \sin^2\theta}\right)$$

$0 \leq \cdots \leq 1$

$$= 2 \quad \text{(by squeeze thm)}$$

So f can be extended to a continuous fcn on $\mathbb{R}^2$ with $f(0,0) = 2$

(5) Let

$$f(x, y, z) = xy \ln(x^2 z^2)$$

(a) (2 points) Find the domain of definition of f.

(b) (3 points) Find all 3 first-order partial derivatives of f.

(c) (9 points) Find all 9 second-order partial derivatives of f.

(d) (2 points) Clairaut's theorem (mixed derivative theorem) reduces the calculation of all 27 third-order partial derivatives of f to the calculation of a smaller number of third-order partial derivatives. What is this smaller number? Justify your answer.

a) Domain of $f = \mathbb{R}^3 \setminus \{x^2 z^2 = 0\}$
$= \{(x, y, z) \in \mathbb{R}^3 : x \neq 0 , z \neq 0\}$

b) $f_x = y \ln(x^2 z^2) + xy \cdot \frac{1}{x^2 z^2} \cdot 2x z^2$
$= y \ln(x^2 z^2) + 2y$

$f_y = x \ln(x^2 z^2)$
$f_z = xy \cdot \frac{1}{x^2 z^2} \cdot 2x^2 z = \frac{2xy}{z}$

c) $f_{xx} = y \cdot \frac{1}{x^2 z^2} \cdot 2x z^2 + 0 = \frac{2y}{x}$
$f_{yy} = 0$
$f_{zz} = -\frac{2xy}{z^2}$
$f_{xy} = \ln(x^2 z^2) + 2 = f_{yx}$ (by Clairaut's)
$f_{xz} = \frac{2y}{z} = f_{zx}$ (by Clairaut's)
$f_{yz} = \frac{x}{z} = f_{zy}$ (by Clairaut's)

(5) Let

$$f(x, y, z) = xy\ln(x^2z^2)$$

(a) (2 points) Find the domain of definition of f.

(b) (3 points) Find all 3 first-order partial derivatives of f.

(c) (9 points) Find all 9 second-order partial derivatives of f.

(d) (2 points) Clairaut's theorem (mixed derivative theorem) reduces the calculation of all 27 third-order partial derivatives of f to the calculation of a smaller number of third-order partial derivatives. What is this smaller number? Justify your answer.

d) By Clairaut's Thm, only need to compute

f_{xxx}, f_{yyy}, f_{zzz}

f_{xxy}, f_{xxz}, f_{xyy}, f_{yyz}, f_{xzz}, f_{yzz}

f_{xyz}

So the smaller number is 10

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(6) Let

$$f(x,y,z) = \begin{cases} \frac{z\sin(x^3+y^4)}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x^2+y^2 = 0. \end{cases}$$

(a) (6 points) Using definition of partial derivatives, find $f_x(0,0,1)$, $f_y(0,0,1)$, and $f_z(0,0,1)$.

(b) (14 points) Using definition of second-order partial derivatives, find $f_{zx}(0,0,1)$ and $f_{zy}(0,0,1)$.

a) $$f_x(0,0,1) = \lim_{h\to 0} \frac{f(h,0,1) - f(0,0,1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin h^3}{h^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\sin h^3}{h^3} = 1$$

$$f_y(0,0,1) = \lim_{h\to 0} \frac{f(0,h,1) - f(0,0,1)}{h}$$

$$= \lim_{h\to 0} \frac{\sin h^4}{h^4} \cdot h = 1 \cdot 0 = 0.$$

$$f_z(0,0,1) = \ldots$$

$$= 0$$

b) ?

$$f_{zx}(0,0,1) = \lim_{h\to 0} \frac{f_z(h,0,1) - f_z(0,0,1)}{h}$$

For $(x,y) \neq (0,0)$,

$$f_z(x,y,z) = \frac{\partial}{\partial z}\left(\frac{z\sin(x^3+y^4)}{x^2+y^2}\right) = \frac{\sin(x^3+y^4)}{x^2+y^2}$$

So $f_z(h,0,1) = \frac{\sin(h^3)}{h^2}$ if $h \neq 0$

(6) Let

$$f(x,y,z) = \begin{cases} \frac{z\sin(x^3+y^4)}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x^2+y^2 = 0. \end{cases}$$

(a) (6 points) Using definition of partial derivatives, find $f_x(0,0,1)$, $f_y(0,0,1)$, and $f_z(0,0,1)$.

(b) (14 points) Using definition of second-order partial derivatives, find $f_{zx}(0,0,1)$ and $f_{zy}(0,0,1)$.

$$f_{zx}(0,0,1) = \lim_{h\to 0} \frac{f_z(h,0,1) - f_z(0,0,1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\sin h^3}{h^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\sin h^3}{h^3} = 1$$

$$f_{zy}(0,0,1) = \cdots$$

$$= 0$$